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Thermodynamics in Modified Gravity Theories

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We demonstrate that there does exist an equilibrium description of thermodynamics on the apparent horizon in the expanding cosmological background for a wide class of modified gravity theories with the Lagrangian density $f(R, \phi, X)$, where R is the Ricci scalar and X is the kinetic energy of a scalar field ϕ . This comes from a suitable definition of an energy momentum tensor of the “dark” component obeying the local energy conservation law in the Jordan frame. It is shown that the equilibrium description in terms of the horizon entropy S is convenient because it takes into account the contribution of the horizon entropy S in non-equilibrium thermodynamics as well as an entropy production term.

Keywords: Modified theories of gravity; Quantum aspects of black holes, evaporation, thermodynamics; Dark energy; Cosmology.

1. Introduction

The discovery of black hole entropy has implied a profound physical connection between gravity and thermodynamics.¹ The gravitational entropy S in the Einstein gravity is proportional to the horizon area A of black holes, such that $S = A/(4G)$, where G is gravitational constant. A black hole with mass M obeys the first law of thermodynamics, $TdS = dM$,² where $T = \kappa_s/(2\pi)$ is a Hawking temperature determined by the surface gravity κ_s .³ Since black hole solutions follow from the Einstein field equations, the first law of black hole thermodynamics implies some connection between thermodynamics and the Einstein equations. In fact, it was shown⁴ that the Einstein equations can be derived by using the Clausius relation $TdS = dQ$ on all local acceleration horizons in the Rindler space-time together with the relation $S \propto A$, where dQ and T are the energy flux across the horizon and

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the Unruh temperature seen by an accelerating observer just inside the horizon, respectively.

In the theories in which the Lagrangian density f is a non-linear function in terms of the Ricci scalar R (so called “ $f(R)$ gravity”), it was pointed out⁵ that a non-equilibrium treatment is required so that the Clausius relation may be modified to $dS = dQ/T + d_i S$, where the horizon entropy S is defined by $S = F(R)A/(4G)$ with $F(R) = \partial f/\partial R$ and $d_i S$ describes a bulk viscosity entropy production term. The variation of the quantity $F(R)$ gives rise to the non-equilibrium term $d_i S$ that is absent in the Einstein gravity. The reason why a non-equilibrium entropy production term $d_i \hat{S}$ appears is closely related to the theories in which the derivative of the Lagrangian density f with respect to R is not constant. It is meaningful to examine whether an equilibrium description of thermodynamics is possible in such modified gravity theories. In the present paper, we review our results⁶ in Ref. 6 and show that equilibrium thermodynamics does exist for the general Lagrangian density $f(R, \phi, X)$ including $f(R)$ gravity and scalar-tensor theories, where f is function of R , a scalar field ϕ , and a field kinetic energy $X = -(\nabla\phi)^2/2$. We mention that in Ref. 7, the derivation in Ref. 4 was also extended to $f(R)$ gravity.

2. Thermodynamics in modified gravity-non-equilibrium picture

We consider the following action

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R, \phi, X) - \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M), \quad (1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, \mathcal{L}_M is the matter Lagrangian that depends on $g_{\mu\nu}$ and matter fields Ψ_M , and $X = -(1/2)g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ is the kinetic term of a scalar field ϕ (∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$). The action (1) can describe a number of modified gravity theories, e.g., $f(R)$ gravity, Brans-Dicke theories, scalar-tensor theories, and dilaton gravity. It also covers scalar field theories such as quintessence and k-essence. From the action (1), the gravitational field equation and the equation of motion for ϕ are derived as

$$F G_{\mu\nu} = 8\pi G T_{\mu\nu}^{(M)} + \frac{1}{2}g_{\mu\nu}(f - RF) + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\square F + \frac{1}{2}f_{,X}\nabla_\mu\phi\nabla_\nu\phi, \quad (2)$$

$$\frac{1}{\sqrt{-g}}\partial_\mu(f_{,X}\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) + f_{,\phi} = 0, \quad (3)$$

where $F \equiv \partial f/\partial R$, $f_{,X} \equiv \partial f/\partial X$, $f_{,\phi} \equiv \partial f/\partial\phi$, $T_{\mu\nu}^{(M)} = (2/\sqrt{-g})\delta\mathcal{L}_M/\delta g^{\mu\nu}$, and $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ is the Einstein tensor. Throughout this paper, we use “ $_Y$ ” for the partial derivative with respect to the variable Y . For the matter energy momentum tensor $T_{\mu\nu}^{(M)}$, we consider perfect fluids of ordinary matter (radiation and non-relativistic matter) with total energy density ρ_f and pressure P_f .

We assume the 4-dimensional Friedmann-Lemaître-Robertson-Walker (FLRW) space-time with the metric, $ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + \bar{r}^2 d\Omega^2$, where $\bar{r} = a(t)r$ and $x^0 = t$, $x^1 = r$ with the 2-dimensional metric $h_{\alpha\beta} = \text{diag}(-1, a^2(t)/[1 - Kr^2])$. Here, $a(t)$

is the scale factor, K is the cosmic curvature, and $d\Omega^2$ is the metric of 2-dimensional sphere with unit radius. In the FLRW background, from Eqs. (2) and (3) we obtain the following field equations:

$$3F(H^2 + K/a^2) = f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + 8\pi G\rho_f, \quad (4)$$

$$-2F(\dot{H} - K/a^2) = f_{,X}X + \ddot{F} - H\dot{F} + 8\pi G(\rho_f + P_f), \quad (5)$$

$$\frac{1}{a^3} \left(a^3 \dot{\phi} f_{,X} \right)' = f_{,\phi}, \quad (6)$$

where $H = \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative of $\partial/\partial t$, and the scalar curvature is given by $R = 6(2H^2 + \dot{H} + K/a^2)$. The perfect fluid satisfies the continuity equation $\dot{\rho}_f + 3H(\rho_f + P_f) = 0$. Equations (4) and (5) can be written as

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3F}(\hat{\rho}_d + \rho_f), \quad \dot{H} - \frac{K}{a^2} = -\frac{4\pi G}{F}(\hat{\rho}_d + \hat{P}_d + \rho_f + P_f), \quad (7)$$

where

$$\hat{\rho}_d \equiv \frac{1}{8\pi G} \left[f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} \right], \quad (8)$$

$$\hat{P}_d \equiv \frac{1}{8\pi G} \left[\ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) \right]. \quad (9)$$

Note that a hat to represent quantities in the non-equilibrium description of thermodynamics. If the density $\hat{\rho}_d$ and the pressure \hat{P}_d of “dark” components are defined in this way, these obey the following equation

$$\dot{\hat{\rho}}_d + 3H(\hat{\rho}_d + \hat{P}_d) = \frac{3}{8\pi G}(H^2 + K/a^2)\dot{F}, \quad (10)$$

where we have used Eq. (6). For the theories with $\dot{F} \neq 0$, the right-hand side (r.h.s.) of Eq. (10) does not vanish, so that the standard continuity equation does not hold. This occurs for $f(R)$ gravity and scalar-tensor theory.

We examine the thermodynamic property of the theories given above. To begin with, the apparent horizon is determined by the condition $h^{\alpha\beta}\partial_\alpha\bar{r}\partial_\beta\bar{r} = 0$, which means that the vector $\nabla\bar{r}$ is null on the surface of the apparent horizon. For the FLRW space-time, the radius of the apparent horizon is given by $\bar{r}_A = (H^2 + K/a^2)^{-1/2}$. In the Einstein gravity, the Bekenstein-Hawking horizon entropy is given by $S = A/(4G)$, where $A = 4\pi\bar{r}_A^2$ is the area of the apparent horizon.^{1,2,3} In the context of modified gravity theories, a horizon entropy \hat{S} associated with a Noether charge was introduced by Wald.^{8,9} The Wald entropy \hat{S} is a local quantity defined in terms of quantities on the bifurcate Killing horizon. More specifically, it depends on the variation of the Lagrangian density of gravitational theories with respect to the Riemann tensor. This is equivalent to $\hat{S} = A/(4G_{\text{eff}})$, where $G_{\text{eff}} = G/F$ is the effective gravitational coupling.¹⁰ By using the Wald entropy

$$\hat{S} = \frac{AF}{4G}, \quad (11)$$

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we obtain

$$\frac{1}{2\pi\bar{r}_A}d\hat{S} = 4\pi\bar{r}_A^3 H(\hat{\rho}_d + \hat{P}_d + \rho_f + P_f)dt + \frac{\bar{r}_A}{2G}dF. \quad (12)$$

The apparent horizon has the following Hawking temperature

$$T = \frac{|\kappa_s|}{2\pi}, \quad \kappa_s = -\frac{1}{\bar{r}_A} \left(1 - \frac{\dot{\bar{r}}_A}{2H\bar{r}_A}\right), \quad (13)$$

where κ_s is $\kappa_s = -(\bar{r}_A/2)(\dot{H} + 2H^2 + K/a^2) = -2\pi G/(3F)\bar{r}_A(\hat{\rho}_T - 3\hat{P}_T)$, with $\hat{\rho}_T \equiv \hat{\rho}_d + \rho_f$ and $\hat{P}_T \equiv \hat{P}_d + P_f$. As long as the total equation of state $w_T = \hat{P}_T/\hat{\rho}_T$ satisfies $w_T \leq 1/3$, we have $\kappa_s \leq 0$, which is the case for standard cosmology. Then, Eq. (12) can be written as

$$Td\hat{S} = 4\pi\bar{r}_A^3 H(\hat{\rho}_d + \hat{P}_d + \rho_f + P_f)dt - 2\pi\bar{r}_A^2(\hat{\rho}_d + \hat{P}_d + \rho_f + P_f)d\bar{r}_A + \frac{T}{G}\pi\bar{r}_A^2 dF. \quad (14)$$

In the Einstein gravity, the Misner-Sharp energy¹¹ is defined to be $E = \bar{r}_A/(2G)$. In $f(R)$ gravity and scalar-tensor theory, this may be extended to the form $\hat{E} = \bar{r}_A F/(2G)$.¹² Using this latter expression for $f(R, \phi, X)$ theories, we find

$$\hat{E} = \frac{\bar{r}_A F}{2G} = V \frac{3F(H^2 + K/a^2)}{8\pi G} = V(\hat{\rho}_d + \rho_f), \quad (15)$$

where $V = 4\pi\bar{r}_A^3/3$ is the volume inside the apparent horizon. It follow from Eq. (15) that

$$d\hat{E} = -4\pi\bar{r}_A^3 H(\hat{\rho}_d + \hat{P}_d + \rho_f + P_f)dt + 4\pi\bar{r}_A^2(\hat{\rho}_d + \rho_f)d\bar{r}_A + \frac{\bar{r}_A}{2G}dF. \quad (16)$$

The combination of Eqs. (14) and (16) gives

$$Td\hat{S} = -d\hat{E} + \hat{W}dV + \frac{\bar{r}_A}{2G}(1 + 2\pi\bar{r}_A T)dF, \quad (17)$$

where we have introduced the work density defined by^{13,14,15} $\hat{W} = (\hat{\rho}_d + \rho_f - \hat{P}_d - P_f)/2$. This equation can be written in the following form:

$$Td\hat{S} + Td_i\hat{S} = -d\hat{E} + \hat{W}dV, \quad (18)$$

where

$$\begin{aligned} d_i\hat{S} &= -\frac{1}{T}\frac{\bar{r}_A}{2G}(1 + 2\pi\bar{r}_A T)dF = -\left(\frac{\hat{E}}{T} + \hat{S}\right)\frac{dF}{F} \\ &= -\frac{\pi}{G}\frac{4H^2 + \dot{H} + 3K/a^2}{(H^2 + K/a^2)(2H^2 + \dot{H} + K/a^2)}dF, \end{aligned} \quad (19)$$

which is consistent with the result¹⁶ of Ref. 16 for $K = 0$ obtained in $f(R)$ gravity and scalar-tensor theories. The new term $d_i\hat{S}$ can be interpreted as a term of entropy production in the non-equilibrium thermodynamics. The theories with $F = \text{constant}$ lead to $d_i\hat{S} = 0$, which means that the first-law of the equilibrium thermodynamics holds. On the other hand, the theories with $dF \neq 0$, including $f(R)$ gravity and scalar-tensor theory, give rise to the additional term (19).

3. Equilibrium description of thermodynamics in modified gravity

The energy density $\hat{\rho}_d$ and the pressure \hat{P}_d defined in Eqs. (8) and (9) do not satisfy the standard continuity equation for $\dot{F} \neq 0$. If it is possible to define $\hat{\rho}_d$ and \hat{P}_d so that they can satisfy the conserved equation, the non-equilibrium description of thermodynamics may not be necessary. In this section, we demonstrate that such a treatment is indeed possible.

One can write Eqs. (4) and (5) as follows:

$$3 \left(H^2 + \frac{K}{a^2} \right) = 8\pi G (\rho_d + \rho_f), \quad -2 \left(\dot{H} - \frac{K}{a^2} \right) = 8\pi G (\rho_d + P_d + \rho_f + P_f), \quad (20)$$

where

$$\rho_d \equiv \frac{1}{8\pi G} \left[f_{,X} X + \frac{1}{2} (FR - f) - 3H\dot{F} + 3(1-F)(H^2 + K/a^2) \right], \quad (21)$$

$$P_d \equiv \frac{1}{8\pi G} \left[\ddot{F} + 2H\dot{F} - \frac{1}{2} (FR - f) - (1-F)(2\dot{H} + 3H^2 + K/a^2) \right]. \quad (22)$$

If we define ρ_d and P_d in this way, they obey the following continuity equation $\dot{\rho}_d + 3H(\rho_d + P_d) = 0$, where we have used Eq. (6). In the equilibrium description, the energy-momentum conservation in terms of “dark” components is met. Since the perfect fluid of ordinary matter also satisfies the continuity equation, the total energy density $\rho_T \equiv \rho_d + \rho_f$ and the total pressure $P_T \equiv P_d + P_f$ of the universe obey the continuity equation $\dot{\rho}_T + 3H(\rho_T + P_T) = 0$. Hence, the equilibrium treatment of thermodynamics can be executed similarly to that in the Einstein gravity. We introduce the Bekenstein-Hawking entropy^{1,2,3} $S = A/(4G) = \pi/[G(H^2 + K/a^2)]$, unlike the Wald entropy. This allows us to obtain the equilibrium description of thermodynamics as that in the Einstein gravity. It follows that

$$\frac{1}{2\pi\bar{r}_A} dS = 4\pi\bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt. \quad (23)$$

Using the horizon temperature in Eq. (13), we get

$$T dS = 4\pi\bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt - 2\pi\bar{r}_A^2 (\rho_d + P_d + \rho_f + P_f) d\bar{r}_A. \quad (24)$$

By defining the Misner-Sharp energy to be $E = \bar{r}_A/(2G) = V(\rho_d + \rho_f)$, we obtain

$$dE = -4\pi\bar{r}_A^3 H (\rho_d + P_d + \rho_f + P_f) dt + 4\pi\bar{r}_A^2 (\rho_d + \rho_f) d\bar{r}_A. \quad (25)$$

Due to the conservation equation for “dark” components, the r.h.s. of Eq. (25) does not include an additional term proportional to dF . Combining Eqs. (24) and (25) gives

$$T dS = -dE + W dV, \quad (26)$$

where the work density W is defined by $W = (\rho_d + \rho_f - P_d - P_f)/2$. Equation (26) corresponds to the first law of equilibrium thermodynamics. This shows that

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the equilibrium form of thermodynamics can be derived by introducing the energy density ρ_d and the pressure P_d in a suitable way. It follows from Eq. (26) that

$$T\dot{S} = V \left(3H - \frac{\dot{V}}{2V} \right) (\rho_d + \rho_f + P_d + P_f). \quad (27)$$

Using $V = 4\pi\bar{r}_A^3/3$ and Eq. (13), we acquire

$$\dot{S} = 6\pi H V \bar{r}_A (\rho_d + \rho_f + P_d + P_f) = -\frac{2\pi}{G} \frac{H(\dot{H} - K/a^2)}{(H^2 + K/a^2)^2}. \quad (28)$$

The horizon entropy increases as long as the null energy condition $\rho_T + P_T = \rho_d + \rho_f + P_d + P_f \geq 0$ is met.

The above equilibrium description of thermodynamics is intimately related with the fact that there exists an energy momentum tensor $T_{\mu\nu}^{(d)}$ satisfying the local conservation law $\nabla^\mu T_{\mu\nu}^{(d)} = 0$. This corresponds to writing the Einstein equation in the form

$$G_{\mu\nu} = 8\pi G \left(T_{\mu\nu}^{(d)} + T_{\mu\nu}^{(M)} \right), \quad (29)$$

where

$$T_{\mu\nu}^{(d)} \equiv \frac{1}{8\pi G} \left[\frac{1}{2} g_{\mu\nu} (f - R) + \nabla_\mu \nabla_\nu F - g_{\mu\nu} \square F + \frac{1}{2} f_{,X} \nabla_\mu \phi \nabla_\nu \phi + (1 - F) R_{\mu\nu} \right]. \quad (30)$$

Defining $T_{\mu\nu}^{(d)}$ in this way, the local conservation of $T_{\mu\nu}^{(d)}$ follows from Eq. (29) because of the relations $\nabla^\mu G_{\mu\nu} = 0$ and $\nabla^\mu T_{\mu\nu}^{(M)} = 0$.

It can be shown that the horizon entropy S in the equilibrium description has the following relation with \hat{S} in the non-equilibrium description:

$$dS = d\hat{S} + d_i \hat{S} + \frac{\bar{r}_A}{2GT} dF - \frac{2\pi(1-F)}{G} \frac{H(\dot{H} - K/a^2)}{(H^2 + K/a^2)^2} dt. \quad (31)$$

By using the relations (19) and (23), Eq. (31) is rewritten to the following form

$$dS = \frac{1}{F} d\hat{S} + \frac{1}{F} \frac{2H^2 + \dot{H} + K/a^2}{4H^2 + \dot{H} + 3K/a^2} d_i \hat{S}, \quad (32)$$

where

$$d_i \hat{S} = -\frac{6\pi}{G} \frac{4H^2 + \dot{H} + 3K/a^2}{H^2 + K/a^2} \frac{dF}{R}. \quad (33)$$

The difference appears in modified gravity theories with $dF \neq 0$, whereas S is identical to \hat{S} in the Einstein gravity ($F = 1$). From Eq. (32), we see that the change of the horizon entropy S in the equilibrium framework involves the information of both $d\hat{S}$ and $d_i \hat{S}$ in the non-equilibrium framework.

As an example, we apply the formulas of the horizon entropies in the Jordan frame to inflation in $f(R)$ theories. In what follows, we assume the flat FLRW space-time ($K = 0$). We consider the model $f(R) = R + \alpha R^n$ ($\alpha, n > 0$) in the region $F = df/dR = 1 + n\alpha R^{n-1} \gg 1$. The first inflation model for $n = 2$ has

been proposed by Starobinsky.¹⁷ During inflation, the approximations $|\dot{H}/H^2| \ll 1$ and $|\ddot{H}/(H\dot{H})| \ll 1$ can be used. With these approximations, in the absence of matter fluids the first equation in (20) is reduced to $\dot{H}/H^2 = -\beta$, where $\beta \equiv (2-n)/[(n-1)(2n-1)]$. From this, we see that the scale factor behaves as $a \propto t^{1/\beta}$. Hence, for $\beta < 1$, i.e. $n > (1 + \sqrt{3})/2$, power-law inflation occurs. If $n = 2$, one obtains $\beta = 0$, so that H is constant in the regime $F \gg 1$. The models with $n > 2$ lead to the super-inflation characterized by $\dot{H} > 0$ and $a \propto |t_0 - t|^{-1/|\beta|}$ (t_0 is a constant). The standard inflation with decreasing H occurs for $0 < \beta < 1$, i.e. $(1 + \sqrt{3})/2 < n < 2$. In this case, the horizon entropy $S = A/(4G) = \pi/(GH^2)$ in the equilibrium framework grows as $S \propto H^{-2} \propto t^2$ during inflation. Meanwhile, the horizon entropy $\hat{S} = F(R)A/(4G)$ in the non-equilibrium framework has a dependence $\hat{S} \propto R^{n-1}/H^2 \propto H^{2(n-2)} \propto t^{2(2-n)}$ in the regime $F \gg 1$. Thus, \hat{S} grows more slowly relative to S . This property can be understood from Eq. (32), i.e.

$$\frac{dS}{dt} = \frac{1}{F} \frac{d\hat{S}}{dt} + \frac{1}{F} \frac{2-\beta}{4-\beta} \frac{d_i \hat{S}}{dt}, \quad (34)$$

where

$$\frac{d_i \hat{S}}{dt} = \frac{12\pi\beta(4-\beta)}{G} H F_{,R}. \quad (35)$$

Here, $F_{,R} \equiv dF/dR$. For the above model, the term $F = n\alpha R^{n-1}$ evolves as $F \propto t^{2(1-n)}$. This means that $(1/F)d\hat{S}/dt \propto t$ in Eq. (34), which has the same dependence as the time-derivative of S , i.e. $dS/dt \propto t$. The r.h.s. of Eq. (35) is positive because $F_{,R} > 0$ and $\beta > 0$, so that $d_i \hat{S}/dt > 0$. We have $d_i \hat{S}/dt \propto t^{3-2n}$ and therefore the last term on the r.h.s. of Eq. (34) also grows in proportion to t . As a consequence, S evolves differently from \hat{S} due to the presence of the term $1/F$.

4. Summary

We have explored thermodynamics on the apparent horizon with area A in the expanding cosmological background for a wide class of modified gravity theories with the Lagrangian density $f(R, \phi, X)$. We have examined both non-equilibrium and equilibrium descriptions of thermodynamics.

In a non-equilibrium description of thermodynamics, the energy density and the pressure of “dark” components are defined such that they cannot satisfy the standard continuity equation for the theories in which the quantity $F = \partial f / \partial R$ is not constant. In addition, the Wald’s horizon entropy in the form $\hat{S} = AF/(4G)$ associated with a Noether charge is introduced, so that a non-equilibrium entropy production term $d_i \hat{S}$ appears. This non-equilibrium description of thermodynamics arises for the theories with $dF \neq 0$, which include $f(R)$ gravity and scalar-tensor theories.

On the other hand, it is possible to acquire an equilibrium description of thermodynamics by defining the energy density and the pressure of “dark” components

so that they can obey the standard continuity equation. In other words, this comes from a suitable definition of an energy momentum tensor of the “dark” component that respects to a local energy conservation law in the Jordan frame. In this framework, the horizon entropy S in equilibrium thermodynamics is equal to the Bekenstein-Hawking entropy in the form $S = A/(4G)$, as in the Einstein gravity.

Moreover, we have found that the variation of S can be expressed in terms of $d\hat{S}$ in the non-equilibrium framework together with the entropy production term $d_i\hat{S}$. It is considered that the equilibrium description of thermodynamics is useful not only to provide the General Relativistic analogue of the horizon entropy irrespective of gravitational theories but also to understand the nonequilibrium thermodynamics deeper in connection with the standard equilibrium framework.

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References

1. J. D. Bekenstein, *Phys. Rev. D* **7** (1973) 2333.
2. J. M. Bardeen, B. Carter and S. W. Hawking, *Commun. Math. Phys.* **31** (1973) 161.
3. S. W. Hawking, *Commun. Math. Phys.* **43** (1975) 199 [Erratum-ibid. **46** (1976) 206].
4. T. Jacobson, *Phys. Rev. Lett.* **75** (1995) 1260.
5. C. Eling, R. Guedens and T. Jacobson, *Phys. Rev. Lett.* **96** (2006) 121301.
6. K. Bamba, C. Q. Geng and S. Tsujikawa, *Phys. Lett. B* **688** (2010) 101 [arXiv:0909.2159 [gr-qc]].
7. E. Elizalde and P. J. Silva, *Phys. Rev. D* **78** (2008) 061501.
8. R. M. Wald, *Phys. Rev. D* **48** (1993) 3427.
9. V. Iyer and R. M. Wald, *Phys. Rev. D* **50** (1994) 846.
10. R. Brustein, D. Gorbonos and M. Hadad, *Phys. Rev. D* **79** (2009) 044025.
11. C. W. Misner and D. H. Sharp, *Phys. Rev.* **136** (1964) B571.
12. Y. Gong and A. Wang, *Phys. Rev. Lett.* **99** (2007) 211301.
13. S. A. Hayward, *Phys. Rev. D* **49** (1994) 6467.
14. S. A. Hayward, *Class. Quant. Grav.* **15** (1998) 3147.
15. S. A. Hayward, S. Mukohyama and M. C. Ashworth, *Phys. Lett. A* **256** (1999) 347.
16. S. F. Wu, B. Wang, G. H. Yang and P. M. Zhang, *Class. Quant. Grav.* **25** (2008) 235018.
17. A. A. Starobinsky, *Phys. Lett. B* **91** (1980) 99.